

ON DISCRETE PROJECTIVE TRANSFORMATION GROUPS OF RIEMANNIAN MANIFOLDS

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ABSTRACT. We prove rigidity facts for groups acting on pseudo-Riemannian manifolds by preserving unparameterized geodesics.

RÉSUMÉ. Nous démontrons des résultats de rigidité pour les groupes agissant sur des variétés pseudo-riemanniennes en préservant leurs géodésiques non-paramétrées.

1. INTRODUCTION

1.0.1. *The projective group of a connection.* Two linear connections ∇ and ∇' on a manifold M are equal iff they have the same (parameterized) geodesics. They are called projectively equivalent if they have the same unparameterized geodesics. This is equivalent to that the difference $(2, 1)$ -tensor $T = \nabla - \nabla'$ being trace free in a natural sense [11].

The affine group $\text{Aff}(M, \nabla)$ is that of transformations preserving ∇ and the projective one $\text{Proj}(M, \nabla)$ is that of transformations f sending ∇ to a projectively equivalent one. So, elements of Aff are those preserving (parameterized) geodesics and those of Proj preserve unparameterized geodesics.

Obviously $\text{Aff} \subset \text{Proj}$; and it is natural to look for special connections for which this inclusion is proper, that is when projective non-affine transformations exist?

1.0.2. *Case of Levi-Civita connections.* Let now g be a Riemannian metric on M and ∇ its Levi-Civita connection. The affine and projective groups $\text{Aff}(M, g)$ and $\text{Proj}(M, g)$ are those associated to ∇ .

More generally, g and g' are projectively equivalent if so is the case for their associated Levi-Civita connections. This defines an equivalence relation on the space $\text{Riem}(M)$ of Riemannian metrics on M . Let $\mathcal{P}(M, g)$ denote the class of g , i.e. the set of metrics shearing the same unparameterized geodesics with g . It contains $\mathbb{R}^+ \cdot g$, the set of constant multiples of g . Generically, $\mathcal{P}(M, g) = \mathbb{R}^+ \cdot g$.

One crucial fact here is that $\mathcal{P}(M, g)$ is always a finite dimensional manifold whose dimension is called the *degree of projective mobility* of g . (This contrasts with the case of projective equivalence classes of connections which are infinitely dimensional affine spaces. Similarly, conformal classes of metrics are identified to spaces of positive functions on the manifold). It is actually one culminate fact of projective differential geometry to identify

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$\mathcal{P}(M, g)$ to an open subset of a finite dimensional linear sub-space $\mathcal{L}(M, g)$ of endomorphisms of TM (see §3). Being projectively equivalent for connections is a linear condition, but this is no longer linear for metrics (say because the correspondence $g \rightarrow$ its Levi-Civita connection, is far from being linear!). The trick is to perform a transform leading to a linear equation, see [5] for a nice exposition.

1.0.3. *Philosophy.* The idea behind our approach here is to let a diffeomorphism f on a differentiable manifold M act on the space $\text{Riem}(M)$ of Riemannian metrics on M . That this action has a fixed point means exactly that f is an isometry for some Riemannian metric on M . One then naturally wonder what is the counterpart of the fact that the f -action preserves some (finite dimensional) manifold $V \subset \text{Riem}(M)$. A classical similar idea is to let the isotopy class of a diffeomorphism on a surface act on its Teichmüller space [31]. Here, as it will be seen below, we are specially concerned with the case $\dim V = 2$.

1.0.4. *More general pseudo-Riemannian framework.* All this generalizes to the pseudo-Riemannian case. One fashion to unify all is to generalize all this to the wider framework of second order ordinary differential equations (e.g. hamiltonian systems) on M , by letting their solutions playing the role of (parameterized) geodesics.

1.0.5. *Rigidity of the projective group.* We are interested here in a (very) natural and classical problem in differential geometry: *Characterize pseudo-Riemannian manifolds (M, g) for which $\text{Proj}(M, g) \supsetneq \text{Aff}(M, g)$, that is M admits an essential projective transformation?* Constructing upon a long research history by many people (see for instance [21, 22]), we dare formulate more precisely:

Projective Lichnerowicz conjecture 1.1. *Let (M, g) be a compact pseudo-Riemannian manifold. Then, unless (M, g) is a finite quotient of the standard Riemannian sphere, $\text{Proj}(M, g)/\text{Aff}(M, g)$ is finite.*

— *Same question when compactness is replaced by completeness (this does not contain the first case since general pseudo-Riemannian non-Riemannian compact manifolds may be non-complete).*

1.0.6. *Gromov's vague conjecture.* It states that rigid geometric structures having a large automorphism group, are classifiable [8, 14]! This needs a precise experimental (realistic) formulation for each geometric structure. Our question above is an optimistic formulation in the case of metric projective connections (those which are of Levi-Civita type). The historical case was that of Riemannian conformal structures with a precise formulation, as in the projective case above with the sphere playing a central role, is generally attributed to Lichnerowicz, and solved by J. Ferrand [12, 26]. In the general conformal pseudo-Riemannian case, there are many “Einstein universes”, i.e. conformally flat examples with an essential conformal group. A Lichnerowicz type conjecture would be that all pseudo-Riemannian manifolds with an essential conformal group are conformally flat. However, this was recently invalidated by C. Frances [13]. In the projective case, there is no natural candidate

of a compact pseudo-Riemannian (non-Riemannian) manifold playing the role Einstein universes; it becomes a natural challenge to prove that indeed Proj/Aff is always finite in this situation?

In the vein of this vague conjecture, it is surely interesting to see to automorphism groups of non-metric projective connections...

1.1. Results. This very classical subject of differential geometry was specially investigated by the Italian and next the Soviet schools. All famous names: Beltrami, Dini, Fubini, Levi-Civita are still involved in results on projective equivalence of metrics [3, 9, 20]. As for the “Soviet” side, let us quote o [1, 2, 27, 21, 22, 28, 29], and as names Solodovnikov who “introduced” the projective group problem, and last V. Matveev, who handled many remarkable cases of it.

1.1.1. Killing fields variant. Actually, it was $\text{Proj}^0(M, g)$, the identity component of $\text{Proj}(M, g)$, that got real interest in the literature. Its elements are those belonging to flows of projective Killing fields. There is a prompt formulation of the Lichnerowicz conjecture here: *if $\text{Proj}^0(M, g) \supsetneq \text{Aff}^0(M, g)$, then (M, g) is covered by the standard sphere* (assuming M compact).

This identity component variant was proved by V. Matveev in the case of Riemannian manifolds [21], and remains open in the case of higher signature.

Local actions, i.e. projective Killing fields with flows defined only locally, were also considered, see for instance [6].

However, situations with no Killing fields involved, say for example when Proj is a discrete group do not seem to be studied. We think it is worthwhile to consider them because the discrete part may have dynamics stronger than the connected one, as in the case of a flat torus \mathbb{T}^n , but in fact for its affine group whose discrete part is the beautiful arithmetic (the best!) group $\text{SL}_n(\mathbb{Z})$.

1.1.2. Non dynamical variant. Without actions, one may think of having big $\mathcal{P}(M, g)$ as an index of symmetry, and one naturally may ask when this happens. For this, as in the projective case, consider $\mathcal{A}(M, g)$, the set of metrics affinely equivalent to g (i.e. having the same Levi-Civita connection). Here, we have the following wonderful theorem:

Theorem 1.2 (Kiosak, Matveev, Mounoud, see [22]). *Let (M, g) be a compact pseudo-Riemannian manifold. If $\dim \mathcal{P}(M, g) \geq 3$, then $\mathcal{P}(M, g) = \mathcal{A}(M, g)$, unless (M, g) is covered by the standard Riemannian sphere. In particular $\text{Proj}(M, g) = \text{Aff}(M, g)$ in this case.*

1.1.3. Rank 1 case? In view of this, it remains to consider the case $\dim \mathcal{P}(M, g) = 2$ (the dimension 1 case is trivial). Actually, this case occupies a large part in proofs of Lichnerowicz conjecture in the Riemannian as well as Kählerian cases [21, 23, 24]. (We think our approach here, besides it treats the discrete case, also simplifies these existing proofs). We are not surprized of the resistance of this case, reminiscent to a rank 1 phenomena, vs the higher rank case. Assuming $\dim \mathcal{P}(M, g) \geq 3$ hides a symmetry abundance hypothesis!

Anyway, in all our proofs, we will assume $\dim \mathcal{P}(M, g) = 2$.

1.1.4. *Aim.* Our first objective here is to provide a proof of the above conjecture in case of compact Riemannian manifolds

Theorem 1.3. *Let (M, g) be a compact Riemannian manifold. If M is not a Riemannian finite quotient of a standard sphere, and $\text{Proj}(M, g) \supsetneq \text{Aff}(M, g)$, then $\text{Proj}(M, g)$ is a finite extension of $\text{Aff}(M, g)$.*

More precisely, $\text{Aff}(M, g) = \text{Iso}(M, g)$, and a subgroup $\text{Iso}'(M, g)$ of index ≤ 2 , is normal in $\text{Proj}(M, g)$, and the quotient group $\text{Proj}(M, g)/\text{Iso}'(M, g)$ is either cyclic of order $\leq \dim M$, or dihedral of order $\leq 2 \dim M$.

Examples. In order to illustrate the non-linear character of projective equivalence, let us recall the Dini's classical result: two metrics on a surface are projectively equivalent, iff, at a generic point, they have the following forms in some coordinate system:

$$g = (X(x) - Y(y))(dx^2 + dy^2), \quad \bar{g} = \left(\frac{1}{Y(y)} - \frac{1}{X(x)}\right)\left(\frac{dx^2}{X(x)} + \frac{dy^2}{Y(y)}\right)$$

It follows that for the metric $g = (a(x) - \frac{1}{a(y)})(\sqrt{a(x)}dx^2 + \frac{1}{\sqrt{a(y)}}dy^2)$, the involution $(x, y) \rightarrow (y, x)$ is projective. This example given by V. Matveev [21], shows that non-affine projective transformations may exist (outside the case of spheres) but are not in the identity component, because of his result. Theorem 1.3 says that the “discrete projective transformation group” is always finite, but we do not know examples more complicated than the last involution.

Remark 1.4. *Some of quoted results are also true in the complete non-compact case, but we consider here compact manifolds, only.*

1.2. **Kähler version.** Let (M, g) be a Hermitian manifold. Let $V \subset M$ be a geodesic surface which is at the same time a holomorphic curve. If g is Kähler, then any (real) curve c in V satisfies that its complexified tangent direction is parallel; it is therefore called h -planar. It is very special that such V exists, but h -planar curves always exist. Two Kähler metrics are h -projectively equivalent if they share the same h -planar curves. A holomorphic diffeomorphism f is h -projective if f_*g is h -projectively equivalent to g . Their group is denoted $\text{Proj}^{\text{Hol}}(M, g)$.

This holomorphic side of the projective transformation problem was classically investigated by the Japanese school [15, 17, 32, 33]. Finally, V. Matveev and S. Rosemann generalize all known Riemannian results (on the identity component) to the Kähler case [23]. Our proof here for full groups also extends to the Kähler case and yields the same statement as in Theorem 1.3, where the role of the sphere is played by the complex projective space $\mathbb{P}^N(\mathbb{C})$ endowed with its Fubini-Study metric g_{SF} .

Projective vs projective. Recall that a complex manifold M is called projective if it is holomorphic to a (closed regular) complex submanifold of some projective space $\mathbb{P}^N(\mathbb{C})$. Endowed with the restriction of g_{SF} , $(M, g_{SF}|_M)$ is a Kähler manifold. However, only few (other) Kähler metrics (M, g) admit holomorphic isometric embedding in a projective space (but, of course real analytic isometric embedding exist, by Nash Theorem). The dramatic example is that of an elliptic curve, that is a 2-torus with a complex structure. It admits a large space of holomorphic embedding in projective spaces of different dimensions, but the

induced metric on them can never be flat! This is one case of a “Theorema Egregium” due to Calabi [7] which says that holomorphic isometric immersions in space forms of constant holomorphic sectional curvature, are absolutely rigid (see §8).

Theorem 1.5. *Let (M^d, g) be a complex submanifold of a projective space $\mathbb{P}^N(\mathbb{C})$ endowed with the induced metric (from the normalized Fubini-Study). Then the group $\text{Proj}^{\text{Hol}}(M, g)$ of holomorphic projective transformations is a finite extension of $\text{Iso}^{\text{Hol}}(M, g)$, its group of holomorphic isometries, unless (M, g) is holomorphically homothetic to $\mathbb{P}^d(\mathbb{C})$. More precisely, up to composition with $\text{SU}(1+N)$, M is the image of a Veronese map: $v_k : \mathbb{P}^d(\mathbb{C}) \rightarrow \mathbb{P}^N(\mathbb{C})$ (which expands the metric by a factor k).*

Remark 1.6. *There are submanifolds $M \subset \mathbb{P}^N(\mathbb{C})$ with a big “projective” group, say such that $G_M = \{g \in \text{GL}_{N+1}(\mathbb{C}), g.M = M\}$ is non-compact and acts transitively on M . So, G_M does not act projectively with respect to the induced metric, unless M is a Veronese submanifold. The G_M -action preserves another kind of geometric structures? It is however remarkable that all the automorphism group of any Kähler manifold preserves a huge class of minimal submanifolds (in the sense of Riemannian geometry), namely, complex submanifolds!*

1.3. Towards the general pseudo-Riemannian case. It was proved in [22] that the quotient space $\text{Proj}^0(M, g)/\text{Aff}^0(M, g)$ has always dimension ≤ 1 . We have the following generalization to full groups.

Theorem 1.7. *Let (M, g) be a compact pseudo-Riemannian manifold having an essential projective group, that is, $\text{Proj}(M, g)/\text{Aff}(M, g)$ is infinite. Then, up to finite index:*

- 1) $\text{Aff}(M, g) = \text{Iso}(M, g)$ and it is a normal subgroup of $\text{Proj}(M, g)$.
- 2) $\text{Proj}(M, g)/\text{Iso}(M, g)$ is isomorphic to a subgroup of \mathbb{R} . More precisely, there is a representation $\text{Proj}(M, g) \rightarrow \text{SL}_2(\mathbb{R})$ whose kernel is $\text{Aff}(M, g)$ and range contained in a 1-parameter group.

1.3.1. Organization. We restrict ourselves here to compact manifolds, and from §4 to the case of metrics of projective mobility $\dim \mathcal{P}(M, g) = 2$.

Our proofs are mostly algebraic, somewhere dynamical but rarely geometrical!

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2. ACTIONS, GENERAL CONSIDERATIONS

M is here a compact smooth manifold.

2.0.2. Let \mathcal{E} be the space of $(1, 1)$ -tensors T , i.e. sections of the linear bundle $\text{End}(TM) \rightarrow M$: for any x , T_x is linear map $T_x M \rightarrow T_x M$. The space \mathcal{E} has a natural structure of algebra with unit element I the identity of TM (over \mathbb{R} as well as over $C^\infty(M)$ or $C^k(M)$).

$\text{Diff}(M)$ acts naturally on \mathcal{E} by $(f, T) \rightarrow \rho^E(f)T = f_*T$ defined naturally by $(f_*P)_x = D_{f^{-1}x} f T_{f^{-1}x} D_x f^{-1}$.

2.0.3. Let \mathcal{G} be the space of pseudo-Riemannian metrics on M . Then, $\text{Diff}(M)$ acts on \mathcal{G} by taking direct image, $(f, g) \rightarrow \rho^G(f)g = f_*g$ defined by $(f_*g)_x(u, v) = g_{f^{-1}(x)}((D_x f)^{-1}u, (D_x f)^{-1}v)$.

2.0.4. *Notation.* We will sometimes use the usual notations f_*T and f_*g for $\rho^E(f)T$ and $\rho^G(f)g$, respectively.

2.0.5. *Transfer.* Given a metric g_0 on M , any other metric g can be written $g(\cdot, \cdot) = g_0(T\cdot, \cdot)$, where the transfer tensor $T = T(g, g_0)$ is a g_0 -symmetric $(1, 1)$ -tensor (i.e. T_x is a symmetric endomorphism of $(T_x M, g_{0x})$).

In fact, a metric g defines a bundle isomorphism $TM \rightarrow T^*M$, and thus $T(g, g_0) = g_0^{-1}g$.

In other words, we have a map $C_{g_0} : g \in \mathcal{G} \rightarrow T(g, g_0) = g_0^{-1}g \in \mathcal{E}$. In particular $C_{g_0}(g_0) = I$ (the identity of TM).

2.0.6. *Transfer action.* The transfer of the natural Diff-action ρ^E on \mathcal{G} to \mathcal{E} by means of C_{g_0} , is by definition

$$(f, T) \rightarrow \rho^{GE}(f)(T) = C_{g_0}(\rho^G(f)(C_{g_0}^{-1}T))$$

It equals:

$$g_0^{-1}\rho^G(f)(g_0T) = g_0^{-1}(\rho^G(f)g_0)(\rho^E(f)T) = S_f\rho^E(f)T$$

where the g_0 -strength of f is $S_f = g_0^{-1}(\rho^G(f)g_0)$.

2.0.7. *A preserved functional.* The following “norm-like” functional $Q(T) = \int_M \sqrt{|\det T|} dv_{g_0}$ is preserved by ρ^{GE} . Indeed,

$$Q(\rho^{GE}(f)T) = \int_M \sqrt{|(\det S_f)\det(f_*T)|} dv_{g_0} = \int_M \sqrt{|\det T|}(f^{-1}(x)) \text{Jac } f^{-1} dv_{g_0},$$

and this equals $Q(T)$.

2.0.8. Consider now the partially defined transform $\mathcal{F} : L \in \mathcal{E} \rightarrow T = \frac{L^{-1}}{\det L} \in \mathcal{E}$. Its inverse map is given by $\mathcal{F}^{-1}(T) = (\det T)^{\frac{1}{1+d}} T^{-1}$ ($d = \dim M$).

It is remarkable that \mathcal{F} commutes with the Diff-action ρ^E on \mathcal{E} . The finite dimensional version of this for a linear space E is that $u \rightarrow \text{End}(E) \rightarrow \frac{u^{-1}}{\det u} \in \text{End}(E)$ commutes with the $\text{GL}(E)$ action given by $(A, u) \in \text{GL}(E) \times \text{End}(E) \rightarrow AuA^{-1} \in \text{End}(E)$

2.0.9. *Action in the L -representation.* Consider now the map

$$g \in \mathcal{G} \rightarrow L = \mathcal{F}^{-1}(C_{g_0}(g)) \in \mathcal{E}$$

In other words, to a metric g , we associate the $(1, 1)$ -tensor L such that $g(\cdot, \cdot) = \frac{1}{\det L} g_0(L^{-1}\cdot, \cdot)$.

The corresponding action ρ on \mathcal{E} is given by:

$$\rho(f)L = (\rho^E(f)L)K_f$$

where K_f , the g_0 -strength of f in the L -representation, is the \mathcal{F}^{-1} -transform of S_f , that is K_f is defined by $\rho^G(f)g_0(\cdot, \cdot) = \frac{1}{\det K_f} g_0(K_f^{-1}\cdot, \cdot)$.

Corresponding to Q , ρ preserves the partially defined functional: $L \rightarrow N(L) = \int_M \frac{1}{\det L^{(1+d)/2}} dv_{g_0}$

2.0.10. *The chain rule for strength.*

$$K_{f^n} = (f_*^{n-1}K_f)(f_*^{n-2}K_f) \dots (f_*K_f)K_f$$

(of course $(f^k)_* = (f_*)^k$).

2.0.11. Summarizing:

Fact 2.1. Let g_0 be a fixed metric on M . To any metric g , let L be the $(1,1)$ tensor defined by $g(\cdot, \cdot) = \frac{1}{\det L} g_0(L^{-1} \cdot, \cdot)$. The Diff-action on $(1,1)$ tensors deduced from the usual action on metrics by means of this map $g \rightarrow L$ is given by

$$(f, L) \in \text{Diff}(M) \times \mathcal{E} \rightarrow \rho(f)L = (f_*L)K_f$$

Here K_f is the L -tensor associated to f_*g , i.e. $f_*g = \frac{1}{\det K_f} g_0(K_f^{-1} \cdot, \cdot)$, and f_*L denotes the usual action on \mathcal{E} .

- f is an isometry of $g_0 \iff K_f = I$
- f is a g_0 -similarity (that is $f_*g_0 = bg_0$ for some constant b) $\iff K_f = bI$ for some b .
- ρ preserves the function $L \rightarrow N(L) = \int_M \frac{1}{\det L^{(1+d)/2}} dv_{g_0}$

3. LINEARIZATION, REPRESENTATION OF $\text{Proj}(M, g)$ IN $\mathcal{L}(M, g)$

(We will henceforth mostly deal with only one metric and so we will denote it g instead of g_0).

3.0.12. The space $\mathcal{L}(M, g)$. Recall that $\mathcal{P}(M, g)$ denotes the class of metrics projectively equivalent to g .

Let $\mathcal{L}^*(M, g)$ be the image of $\mathcal{P}(M, g)$ under the correspondence of Fact 2.1, and $\mathcal{L}(M, g)$ its linear hull:

$$\mathcal{L}(M, g) = \{L = \sum_i a_i L_i, a_i \in \mathbb{R}, \text{ such that } \frac{1}{\det L_i} g(L_i^{-1} \cdot, \cdot) \text{ is projectively equivalent to } g\}$$

Let us call \mathcal{L} -tensors the elements of this space.

3.0.13. Linearization.

Theorem 3.1. [5, 27] $L \in \mathcal{L}(M, g)$ iff L satisfies the linear equation:

$$g((\nabla_u L)v, w) = \frac{1}{2} g(v, u) d\text{trace}(L)(w) + \frac{1}{2} g(w, u) d\text{trace}(L)(v)$$

where ∇ is the Levi-Civita connection of g .

Furthermore:

- $\mathcal{L}^*(M, g)$ is an open subset of $\mathcal{L}(M, g)$: an element $L \in \mathcal{L}(M, g)$ belongs to $\mathcal{L}^*(M, g)$ iff it is an isomorphism of TM .
- $\mathcal{L}(M, g)$ has finite dimension (bounded by that corresponding to the projective space of same dimension).
- $L \in \mathcal{L}^*(M, g)$ is parallel iff the corresponding metric $\frac{1}{\det L} g(L^{-1} \cdot, \cdot)$ is affinely equivalent to g , iff L has constant eigenvalues.

3.0.14. Linear representation of $\text{Proj}(M, g)$.

Fact 3.2. We have a finite dimensional representation,

$$f \in \text{Proj}(M, g) \rightarrow \rho(f) \in \text{GL}(\mathcal{L}(M, g))$$

where $\rho(f)(L) = f_*(L) \cdot K_f$.

- ρ preserves the norm-like function $N(L) = \int_M \frac{1}{\det L^{(1+d)/2}} dv_g$.

- Let $p : \text{GL}(\mathcal{B}(M, g)) \rightarrow \text{PGL}(\mathcal{B}(M, g))$ be the canonical projection, then p is injective on $\rho(\text{Proj}(M, g))$, or has at most a kernel $\cong \mathbb{Z}/2\mathbb{Z}$.
- Let \mathcal{D} the subset of degenerate tensors in $\mathcal{L}(M, g)$:

$$\mathcal{D} = \{L \in \mathcal{L}(M, g), L \text{ not an isomorphism of } TM\}$$

Then \mathcal{D} is a closed cone invariant under ρ .

Proof.

- Let aA and A in $\text{GL}(\mathcal{L}(M, g))$ such that both preserve N , then $N(aA(L)) = N(A(L)) = N(L)$, for any L . but $N(aL) = |a|^s N(L)$ with $s = -d(d+1)/2$, and hence $a = \pm 1$.
- $L \in \mathcal{D}$ iff for some $x \in M$, $\det L(x) = 0$. But $\rho(f)L = f_* L K_f$, and hence $\det(\rho(f)L)(f(x)) = \det L(x) \det K_f(x) = 0$

□

Remark 3.3. Actually \mathcal{D} coincides essentially with the ∞ -level of N .

4. THE CASE $\dim \mathcal{L}(M, g) = 2$, A HOMOGRAPHY

4.0.15. *Hypothesis.* Henceforth, we will assume that $\dim \mathcal{L}(M, g) = 2$.

Fix f that is not homothetic, i.e. $K = K_f$ is not a multiple of I . Hence $\mathcal{L}(M, g)$ is spanned by K and I .

4.1. The degenerate set \mathcal{D} .

Fact 4.1. *We have:*

$$\mathcal{D} = \{a(K - tI), a \in \mathbb{R}, \text{ and } t \text{ a real spectral value of } K : \det(K(x) - tI) = 0 \text{ for some } x\}$$

In particular I and $K_f \notin \mathcal{D}$.

If the spectrum is real and described by d continuous eigenfunctions $x \rightarrow \lambda_1(x) \leq \dots \leq \lambda_d(x)$ ($d = \dim M$), then $\mathcal{D} = \cup_{i=1}^{i=d} (C_i \cup -C_i)$, where

$$C_i = \{a(K - tI), a \in \mathbb{R}^+, \text{ and } \inf \lambda_i \leq t \leq \sup \lambda_i\}$$

Each C_i is a proper convex cone (sector).

Finally, unifying intersecting sectors, we get a minimal union: $\mathcal{D} = \cup_{i=1}^{i=k} (D_i \cup -D_i)$, where the D_i are disjoint sectors.

Proof. I (as well as K) do not belong to \mathcal{D} and hence any element of this set has the form $a(K - tI)$. This belongs to \mathcal{D} iff $\det(K(x) - tI) = 0$, for some x , that is $t \in \cup_I (\text{Image}(\lambda_i))$, and the cones C_i follow.

□

4.2. Action by homography.

4.2.1. *Equation.* By the 2-dimensional assumption, there exist α, β such that:

$$\rho(f)(K) = (f_*K)K = \alpha K + \beta I$$

Equivalently,

$$f_*K = \alpha I + \beta K^{-1}$$

Say somehow formally, $f_*K = \frac{\alpha K + \beta I}{K}$.

Since $f_*I = I$, in the basis $\{K, I\}$, $\rho(f) : L \rightarrow f_*(L) \cdot K_f$ has a matrix

$$B = B_f = \begin{pmatrix} \alpha & 1 \\ \beta & 0 \end{pmatrix}$$

4.2.2. The group $\mathrm{GL}_2(\mathbb{R})$ (more faithfully $\mathrm{PGL}_2(\mathbb{R})$) acts on the (projective) circle $\mathbb{S}^1 = \mathbb{R} = \mathbb{R} \cup \infty$, by means of the law

$$z \rightarrow A \cdot z = \frac{az+b}{cz+d} \text{ for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$$

In fact, we can also let $\mathrm{GL}_2(\mathbb{R})$ act on the space of $(1,1)$ -tensors by the same formula: $(A \cdot X)(x) = (aX(x) + bI)(cX(x) + dI)^{-1}$. In other words, the action is fiberwise, and when a fiber $\mathrm{End}(T_x M)$ is identified to $\mathrm{Mat}_n(\mathbb{R})$, then $A \cdot X = \frac{aX+b}{cX+d}$

Now, the previous equation $f_*K = \frac{\alpha K + \beta I}{K}$ can be interpreted by that the linear f_* -action on K equals the homographic action $A \cdot K$ where $A = \begin{pmatrix} \alpha & \beta \\ 1 & 0 \end{pmatrix}$ is the transpose of B .

4.2.3. *Iteration.* We have:

$$f_*^n K = A^n \cdot K, \text{ for any } n \in \mathbb{Z} \quad (4.1)$$

This can be proved in a formal way. Let C be an endomorphism on an abstract algebra $\{1, x, x^{-1}, \dots\}$, such that $C(x) = \alpha + \beta x^{-1}$ (with $C(1) = 1$ and $C(x^{-1}) = C(x)^{-1}$). Then, $C^n(x) = A^n \cdot x$, where $A = \begin{pmatrix} \alpha & \beta \\ 1 & 0 \end{pmatrix}$.

4.2.4. *Significance for eigen-functions.* Let $x \in M$ and $y = f(x)$ and denote $T = D_x f : T_x M \rightarrow T_y M$.

The relation $f_*K = \alpha I + \beta K^{-1}$ means that $T^{-1}K_y T = \alpha + \beta K_x^{-1}$. This implies in particular that T maps an invariant subspace of $(T_x M, K_x)$ to an invariant subspace of $(T_y M, K_y)$. If $E_\lambda(x) \subset T_x M$ is the (generalized) K_x -eigenspace associated to λ , then T maps it to $E_{A \cdot \lambda}(y) \subset T_y M$.

Let $x \rightarrow Sp(x) \subset \mathbb{C}$ be the multivalued spectrum function of K , that is $Sp(x) \subset \mathbb{C}$ is the set of eigenvalues of K_x . Then the image $A \cdot Sp(f(x))$ (of the subset $Sp(f(x))$ under the homography $A^{-1} \cdot$) equals $Sp(x)$, and so

$$Sp(f(x)) = A^{-1} \cdot Sp(x)$$

Also, if $\lambda : M \rightarrow \mathbb{C}$ is a continuous K -eigen-function, that is $\lambda(x) \in Sp(x)$ for any $x \in M$ and λ is continuous, then

$$x \rightarrow \lambda'(x) = A^{-1} \cdot \lambda(f^{-1}(x))$$

is another continuous K -eigen-function.

4.3. Classification of elements of $\mathrm{SL}_2(\mathbb{R})$. Recall that non trivial elements A of $\mathrm{SL}_2(\mathbb{R})$ split into three classes:

- (1) Elliptic: A is conjugate in $\mathrm{SL}_2(\mathbb{R})$ to a rotation, i.e. an element of $\mathrm{SO}(2)$. Its homographic action on $\tilde{\mathbb{R}} = \mathbb{R} \cup \infty$, as well as on $\tilde{\mathbb{C}} = \mathbb{C} \cup \infty$ is conjugate to a rotation.
- (2) Parabolic: A is unipotent, i.e. $(A - 1)$ is nilpotent. Its homographic action on $\tilde{\mathbb{R}}$ as well as on $\tilde{\mathbb{C}}$ is conjugate to a translation. It has a unique fixed point $F_A \in \tilde{\mathbb{R}}$. Up to conjugacy $F_A = \infty$, and $A \cdot z \rightarrow z + a$, where $a \in \mathbb{R}$.

It follows that if $C \subset \mathbb{R}$ is a bounded A -invariant set, then $C = \{F_A\}$ (and necessarily $F_A \neq \infty$).

- (3) Hyperbolic: A has two fixed points F_A^- and F_A^+ . Up to conjugacy, $F_A^- = 0$, $F_A^+ = \infty$, and $f(x) = ax$, with $0 < a < 1$.

Now, if $A \in \mathrm{GL}_2^+(\mathbb{R})$, its homographic action coincides with that of $\frac{A}{\det A}$, and the same classification applies.

5. RIEMANNIAN METRICS: NON-HYPERBOLIC CASES

(M, g) is here a compact Riemannian manifold, and f as in the previous section a chosen element such that $\{K_f, I\}$ generate $\mathcal{L}(M, g)$, and f is not affine.

In this Riemannian setting, all elements of $\mathcal{L}(M, g)$ are diagonalizable (since self-adjoint).

Let $G = \rho(\mathrm{Proj}(M, g))$, and $G^+ = G \cap \mathrm{GL}_2^+(\mathbb{R})$.

Let $A = \rho(h) \in G^+$ be non-trivial, then K_h is not collinear to I . Indeed, assume $K_h = aI$, then recall that $h_*g(\cdot, \cdot) = g(S_h \cdot, \cdot)$ and $S_h = \frac{K_h^{-1}}{\det K_h}$. But, S_h is positive, and finiteness of the volume implies $S_h = I$, and hence also $K_h = I$ (that is $h \in \mathrm{Iso}(M, g)$). Thus, $\rho(h)I = I$. On the other hand, by Fact 4.1, $\rho(h)$ preserves a finite set of lines, all different from $\mathbb{R}I$. Let l_1^+ and l_2^+ be the two nearest half lines to \mathbb{R}^+I . If $\rho(h) \neq Id$, then necessarily $\rho(h)l_1^+ = l_2^+$, but this implies $\rho(h)$ is a reflection which contradicts our hypothesis $\det \rho(h) > 0$.

5.0.1. G^+ can not contain parabolic elements. If $\rho(h)$ is parabolic with fixed point F_h . Then, F_h is the unique real spectral value of K_h (because there is no other bounded set of \mathbb{R} invariant under the associated homography), and thus K_h is proportional to I , which we have just proved to be impossible.

5.0.2. Case where all elements of G^+ are elliptic. Recall that we have a union of $k \leq \dim M$ disjoint sectors D_i , such that $\mathcal{D} = \bigcup_{i=1}^k (D_i \cup -D_i)$ is G -invariant (Fact 4.1). If $k > 1$, then the stabilizer of \mathcal{D} in $\mathrm{SL}_2(\mathbb{R})$ is compact, and we can assume G is a subgroup of $\mathrm{O}(2)$. Now, if a rotation (in $\mathrm{SO}(2)$) preserves a set of k -disjoint sectors, it has order $\leq 2k$.

However, in our case, we know that any $\rho(h) \in G^+$ is $\neq -Id$ (since otherwise $-I = \rho(h)I = h_*(I)K_h = K_h$, and hence $K_h = -I$ which we have already excluded). Say, in other words we can see the rotation acting on the projective space rather than the circle and get exactly k -sectors and deduce that actually, G has order $\leq k$.

As for G if (strictly bigger than G^+), it is dihedral of order $\leq 2k$.

Finally, in the case $k = 1$, that is $\mathcal{D} = D_1 \cup -D_1$, its stabilizer in $\mathrm{SL}_2(\mathbb{R})$ contains $-Id$ together with a one parameter hyperbolic group. So, if we assume all elements of G^+ elliptic, we get $G^+ = \{1\}$. In this case, G itself reduces to a single reflection (if non-trivial).

5.0.3. *About $\text{Iso}(M, g)$.* Observe first that if $h \in \text{Aff}(M, g)$, then necessarily K_h is proportional to I since otherwise K_h will be a combination with constant coefficients of I and K_h , and thus has constant eigenvalues, and therefore $f \in \text{Aff}(M, g)$ contradicting our hypothesis. As observed previously $K_h = I$, that is $h \in \text{Iso}(M, g)$ and so $\text{Aff}(M, g) = \text{Iso}(M, g)$.

On the other hand, if $h \in \text{Iso}(M, g)$, and $\rho(h) \in G^+$, then $\rho(h) = Id$.

In general, if $\rho(h) \neq Id$, then it is a reflection since $\rho(h^2) = Id$.

Let $\text{Iso}^{(2)}(M, g)$ be the normal subgroup of $\text{Iso}(M, g)$ generated by squares $h^2, h \in \text{Iso}(M, g)$.

Then:

- either $\ker \rho = \text{Iso}(M, g)$,
- or $\ker \rho = \text{Iso}^{(2)}(M, g)$, and this has index 2 in $\text{Iso}(M, g)$.
- in all cases, $\text{Iso}(M, g)$ or $\text{Iso}^{(2)}(M, g)$ is normal in $\text{Proj}(M, g)$, and the corresponding quotient is cyclic of order $\leq \dim M$, or dihedral of order $\leq 2 \dim M$.

6. RIEMANNIAN METRICS, HYPERBOLIC CASE

In the present section, (M, g) is a compact Riemannian manifold with $\dim \mathcal{L}(M, g) = 2$ and $f \in \text{Proj}(M, g)$ is such that $\rho(f)$ is hyperbolic.

The final goal (of the section) is to prove that (M, g) is projectively flat.

6.1. Size of the spectrum.

Fact 6.1. *The homography A defined by $\rho(f)$ has two real finite fixed points $\lambda_- < \lambda_+$.*

$K = K_f$ has exactly one non-constant eigen-function λ . It has multiplicity 1 (at generic points), range the interval $[\lambda_-, \lambda_+]$, and satisfies the equivariance: $\lambda(f(x)) = A^{-1} \cdot \lambda_1(x)$.

The full spectrum of K may be $\{\lambda_-, \lambda, \lambda_+\}$, $\{\lambda_-, \lambda\}$ or $\{\lambda, \lambda_+\}$. We denote the multiplicities of λ_- and λ_+ by d_- and d_+ respectively, and hence $\dim M = 1 + d_- + d_+$.

Proof. Let $\mu_1(x) \leq \dots \leq \mu_d(x)$ be the eigenfunctions (with multiplicity) of $K(x)$. From 4.2.4, the map $\mu'_i : x \rightarrow A^{-1} \cdot \mu_i(f^{-1}(x))$ is another eigenfunction and hence equals some μ_j . Taking a power of f , we can assume $\mu'_i = \mu_i$, that is $\mu_i(f^{-1}(x)) = A \cdot \mu_i(x)$. In other words, μ_i is an equivariant map between the two systems (M, f) and $(\mathbb{R}, A^{-1} \cdot)$. Thus, $\text{Image}(\mu_i)$ is a bounded A^{-1} -invariant interval. Hence λ_{\pm} belong to \mathbb{R} (rather than \mathbb{R}) and the image of μ_i can be $\{\lambda_-\}$, $\{\lambda_+\}$ or $[\lambda_-, \lambda_+]$. The fact that only one μ_i has range $[\lambda_-, \lambda_+]$ follows from the following nice fact: :

□

Theorem 6.2. [25] *Let (M, g) be a complete Riemannian manifold and $L \in \mathcal{L}(M, g_0)$. Then two eigen-functions $\mu_i \leq \mu_j$ satisfy $\sup \mu_i \leq \inf \mu_j$ (that is not only $\mu_i(x) \leq \mu_j(x)$, but even $\mu_i(x) \leq \mu_j(y)$ for any $x, y \in M$).*

6.2. Dynamics of f .

Define the singular sets $S_{\pm} = \{x \in M, \lambda(x) = \lambda_{\pm}\}$.

6.2.1. *Lyapunov splitting.* On $M - (S_- \cup S_+)$, corresponding to the eigenspace decomposition of $K = K_f$, we have a regular and orthogonal splitting $TM = E_- \oplus E_+ \oplus E_{\lambda}$.

Due to the relation, $f_*K = \alpha I + \beta K^{-1}$, f preserves this splitting.

Remark 6.3. *Even in the linear situation of a matrix $A \in \text{GL}_d(\mathbb{R})$, it is rare that A^*A and its conjugate $A^{-1}(A^*A)A$ have the same eigenspace decomposition!*

6.2.2. *Distortion.* Recall the definition of the L -strength $f_*g(.,.) = \frac{1}{\det K}g(K^{-1}.,.)$ vs the ordinary strength $S = \frac{L^{-1}}{\det L}$.

If $y = f(x)$ and $u \in T_y M$ belongs to a $\mu = \mu^S(y)$ -eigenspace of S , then $g_x(D_y f^{-1}u, D_y f^{-1}u) = \mu g_y(u, u)$. In our case, $D_x f$ sends S -eigenspaces at x to S -eigenspaces at y , by applying a similarity of ratio $\frac{1}{\sqrt{\mu^S(y)}}$.

In order to compute this by means of K -eigenvalues, observe that

$$\det K(x) = \lambda_-^{d_-} \lambda_+^{d_+} \lambda(x)$$

(where d_-, d_+ are the respective dimension of E_- and E_+).

Thus, for any x , $D_x f$ maps similarly $E_-(x)$ to $E_-(f(x))$ with similarity ratio $\zeta_-(x)$ such that

$$\zeta_-^2(x) = (\det K(f(x))\lambda_- = (\lambda_-^{d_-} \lambda_+^{d_+} \lambda(f(x))\lambda_-$$

As for $D_x f : E_+(x) \rightarrow E_+(f(x))$ and $D_x f : E_\lambda(x) \rightarrow E_\lambda(f(x))$, their respective distortions are:

$$\zeta_+^2(x) = (\lambda_-^{d_-} \lambda_+^{d_+} \lambda(f(x)))\lambda_+ \quad \text{and} \quad \zeta_\lambda^2(x) = (\lambda_-^{d_-} \lambda_+^{d_+} \lambda(f(x)))\lambda(f(x))$$

6.2.3. *Data for f^{-1} .* Let $\lambda_1^*, \lambda_-^*, \lambda_+^*, \zeta_\lambda^*, \dots$ be the analogous quantities corresponding to f^{-1} . Observe that f^{-1} preserves the same Lyapunov splitting and thus $\zeta_-^*(x)\zeta_-(f^{-1}(x)) = 1$. It follows that $\lambda_1^*(x) = \frac{1}{\lambda_1(f^{-1}(x))}$, $\lambda_-^* = \frac{1}{\lambda_+}$, and $\lambda_+^* = \frac{1}{\lambda_-}$.

6.2.4. *Estimation of the Jacobian.* From above we infer that:

$$(\text{Jac } f_x)^2 = (\det D_x f)^2 = (\zeta_-(x)\zeta_\lambda(x)\zeta_+(x))^2 = (\lambda_-^{d_-} \lambda_+^{d_+} \lambda(f(x)))^{1+d}$$

Now, in general, $\text{Jac } f_x^n = \text{Jac } f_{f^{n-1}x} \dots \text{Jac } f_x$, and hence

$$(\text{Jac } f_x^n)^2 = ((\lambda_-^{d_-} \lambda_+^{d_+})^n (\lambda(f^n(x)) \dots \lambda(f(x))))^{1+d}$$

Fact 6.4. *Assume that $A^{-1} \cdot$ is decreasing on $[\lambda_-, \lambda_+]$, that is λ_+ is repulsing and λ_- is attracting, equivalently, λ_1 is decreasing along f -orbits: $\lambda_1(f(x)) \leq \lambda_1(x)$. Then $(\text{Jac } f_x^n)^2$ is uniformly equivalent to $(\lambda_-^{d_-+1} \lambda_+^{d_+})^{n(1+d)}$, when $n \rightarrow +\infty$.*

The proof bases on the relation $\lambda(f^k(x)) = (A^{-1})^k \cdot \lambda(x)$ and the next lemma.

Lemma 6.5. *Let C be a hyperbolic element of $\text{SL}_2(\mathbb{R})$ with fixed points $\lambda_- < \lambda_+$, with λ_- attracting. The sequence*

$$\frac{(C^n \cdot z)(C^{n-1} \cdot z) \dots (C \cdot z)}{\lambda_-^n}$$

converges simply in $[\lambda_-, \lambda_+]$ to a continuous function. The convergence is uniform in any compact subset of $[\lambda_-, \lambda_+]$.

Proof. In a small neighbourhood of λ_- , the C -action is equivalent to a linear contraction fixing λ_- , $h : z \rightarrow \alpha(z - \lambda_-) + \lambda_-$, with $0 < \alpha < 1$. This equivalence is valid also on any compact interval $[\lambda_-, \lambda_+ - \varepsilon]$. Thus $h^n z = \alpha^n(z - \lambda_-) + \lambda_-$. The above product is $(c\alpha^n + 1)(c\alpha^{n-1} + 1) \dots (c\alpha + 1)(c + 1)$, where $c = \frac{z - \lambda_-}{\lambda_-}$. This product is convergent since it can be bounded by $\prod_{i=0}^{n-1} (e^{c|\alpha^i|}) \leq e^{c|\Sigma \alpha^i|}$. \square

Corollary 6.6. *Keep the assumption A^{-1} . decreasing. Then $(\lambda_-^{d_-} \lambda_+^{d_+}) \lambda_- \leq 1$ and $(\lambda_-^{d_-} \lambda_+^{d_+}) \lambda_+ \geq 1$. (In particular $\lambda_- < 1 < \lambda_+$).*

Proof. By the fact above, if $\lambda_-^{d_-+1} \lambda_+^{d_+} > 1$, then $\int_{M-S_-} \text{Jac } f_x^n \rightarrow \infty$ when $n \rightarrow +\infty$ contradicting that M has a finite volume.

The other inequality holds by applying the previous fact to f^{-1} . Observe for this, that indeed the eigen-function λ_1^* corresponding to f^{-1} verifies the same decreasing hypothesis. \square

6.2.5. Justification of the decreasing hypothesis for A^{-1} . Let us see what happens if A^{-1} was increasing in $[\lambda_-, \lambda_+]$. In this case, the volume estimate would give $(\lambda_-^{d_-} \lambda_+^{d_+}) \lambda_+ \leq 1$ and $(\lambda_-^{d_-} \lambda_+^{d_+}) \lambda_- \geq 1$, which leads to the contradiction $\lambda_+ \leq \lambda_-$.

6.3. The projective Weyl tensor. This is a $(3, 1)$ -tensor $W : TM \times TM \times TM \rightarrow TM$, that is invariant under $\text{Proj}(M, g)$, $f : W(D_x f u, D_x f v, D_x f w) = D_x f W(u, v, w)$. In dimension ≥ 3 , its vanishing is the obstruction to projective flatness of (M, g) , that is the fact that (M, g) has a constant sectional curvature.

Unlike the conformal case, the Weyl tensor in the projective case is not a curvature tensor, that is it does not satisfy all the usual symmetries of Riemann curvature tensor (see for instance [4, 11] for more information). Its true definition is as follows. If u, v, w, z are four vectors in $T_x M$ such that any two of them are either equal or orthogonal (that is they are part of an orthonormal basis), then:

$$g_x(W(u, v, w), z) = g_x(R(u, v)w, z) - \frac{1}{n-1}(\delta_v^z \text{Ric}(w, u) - \delta_u^z \text{Ric}(w, v))$$

where Ric is the Ricci tensor and δ is the Kronecker symbol.

6.3.1. Boundedness. By compactness, W is bounded by means of g , that is $\|W(u, v, w)\| \leq C \|u\| \|v\| \|w\|$, for some constant C where $\|\cdot\|$ is the norm associated to g .

6.3.2. Asymptotic growth under Df .

$D_x f^n$ maps similarly $E_-(x)$ to $E_-(f^n(x))$ with a contraction factor

$$\zeta_-(n, x) = \zeta_-(x) \zeta_-(f(x)) \dots \zeta_-(f^{n-1}(x))$$

Say, if $u \in E_-(x)$, then $D_x f^n u = \zeta_-(n, x) u_n$, where $u_n \in E_-(f^n x)$, and $\|u_n\| = \|u\|$. Recall that $\zeta_-(x) = ((\lambda_-^{d_-} \lambda_+^{d_+} \lambda(f(x)) \lambda_-)^{1/2})$. If $x \in M - S_+$, i.e. $\lambda(x) < \lambda_+$, then, by Lemma 6.5, $\zeta_-(n, x)$ grows as

$$\zeta_-(n, x) \approx (\lambda_-^{d_-+2} \lambda_+^{d_+})^{n/2}, \quad n \rightarrow +\infty$$

On defines in the same way $\zeta_+(n, x)$ and $\zeta_\lambda(n, x)$ and gets:

$$\zeta_\lambda(n, x) \approx (\lambda_-^{d_-+2} \lambda_+^{d_+})^{n/2}, \quad \zeta_+(n, x) \approx (\lambda_-^{d_-+1} \lambda_+^{d_+})^{n/2} \approx \zeta_-(n, x) \quad n \rightarrow +\infty$$

Surely, $\zeta_-(n, x)$ is exponentially decreasing.

6.3.3. A first vanishing.

Fact 6.7. *For any x , $W(u, v, w) = 0$ once all u, v, w belong to $E_-(x) \oplus E_\lambda(x)$, or all belong to $E_+(x) \oplus E_\lambda(x)$.*

Proof. If $u, v, w \in E_-(x)$, then

$$\|W(D_x f^n u, D_x f^n v, D_x f^n w)\| \leq (\lambda_-^{d_-+2} \lambda_+^{d_+})^{3n/2} C \|u\| \|v\| \|w\| \approx \zeta_-(n, x)^3 \|u\|$$

However, $z_n = W(D_x f^n u, D_x f^n v, D_x f^n w)$ is nothing but $D_x f^n z$, for $z = W(u, v, w)$. It collapses as $\zeta_-(n, x)^3$, but the maximal collapsing rate of non zero vectors in $T_x M$ is $\zeta_-(n, x)$, and hence $z = W(u, v, w) = 0$. The same estimation apply to $E_- \oplus E_\lambda$ and yields vanishing of W on it. Application of f^{-1} yields vanishing of W on $E_+ \oplus E_\lambda$. \square

6.3.4. Commutation. It was observed in particular in [18] that any $L \in \mathcal{L}(M, g)$ commutes with the Ricci curvature Ric . From it we deduce that $Ric(u, v) = 0$ when u and v belong to two different eigenspaces of L . More precisely $Ric(u, v) = 0$ when $u \in E_\lambda(x)$ and $v \in E_\pm(x)$, or $u \in E_-(x)$ and $v \in E_+(x)$.

6.3.5. A second vanishing.

Fact 6.8. *Let $u \in E_-(x)$ and $v, w \in E_+(x) \oplus E_\lambda(x)$, then $W(u, v, w) \in E_-(x)$.*

Proof. We have to prove that $g_x(W(u, v, w), z) = 0$, whenever $z \in E_+(x) \oplus E_\lambda(x)$.

Observe that $\delta_u^z = 0$ and $Ric(w, u) = 0$, and thus $g_x(W(u, v, w), z) = g_x(R(u, v)w, z)$

On the other hand $g_x(W(w, z, v), u) = g_x(R(w, z)v, u)$ because $\delta_w^u = \delta_z^u = 0$.

Hence $g_x(W(u, v, w), z) = g_x(W(w, z, v), u)$. But, we already know that $W(w, z, v) = 0$ since all w, z, v belong to $E_+(x) \oplus E_\lambda(x)$. \square

Fact 6.9. *In the same conditions $W(u, v, w) = 0$.*

Proof. We know that $W(u, v, w)$ belongs to E_- , say $z = W(u, v, w)$. We can write: $D_x f^n z = \zeta_-(n, x) z_n$, $D_x f^n u = \zeta_-(n, x) u_n$, where $\|z_n\| = \|z\|$ and $\|u_n\| = \|u\|$.

Thus $z_n = W(u_n, D_x f^n v, D_x f^n w)$. But $D_x f^n v$ and $D_x f^n w$ tend to 0 when $n \rightarrow -\infty$, hence $\|z\| = \|z_n\| \rightarrow 0$, that is $z = 0$. \square

6.3.6. Full vanishing.

Similar to the above situation in Fact 6.8, we consider the case where $u, v \in E_+(x) \oplus E_\lambda(x)$ and $w \in E_-(x)$ and prove that $W(u, v, w) \in E_-(x)$.

For this, observe that $Ric(w, u) = Ric(w, v) = 0$, and hence $g_x(W(u, v, w), z) = g_x(R(u, v)w, z)$.

On the other hand, $g_x(W(u, v, z), w) = g_x(R(u, v)z, w)$, since $\delta_u^w = \delta_v^w = 0$. Next, $u, v, z \in E_+(x) \oplus E_\lambda(x)$ and thus $W(u, v, z) = 0$, and consequently $g_x(W(u, v, w), z) = 0$ as claimed.

Now the proof of Fact 6.9 applies here and yields that $W(u, v, w) = 0$.

So, we have proved that $W(u, v, w) = 0$ whenever (at least) two of them are in $E_+(x) \oplus E_\lambda(x)$.

Obviously, we can switch roles of E_+ and E_- , and so get that $W(u, v, w) = 0$ in all cases. \square

7. RIEMANNIAN CASE, PROOF THEOREM 1.3

As previously, (M, g) is a compact riemannian manifold with $\text{Proj}(M, g) \supsetneq \text{Aff}(M, g)$. By [21], the degree of projective mobility $\dim \mathcal{L}(M, g)$ equals 2. Pick $f \in \text{Proj}(M, g)$ as in §4.

If $\rho(f)$ is non-hyperbolic for any choice of such f , then, by §5, $\text{Proj}(M, g)/\text{Iso}(M, g)$ is finite as stated in Theorem 1.3.

If $\rho(f)$ is hyperbolic, then by §6, the projective Weyl tensor of (M, g) vanishes. In dimension ≥ 3 , this means that (M, g) has constant sectional curvature. The universal cover of M is necessarily the sphere since the Euclidean and hyperbolic spaces have no projective non-affine transformations.

It remains finally to consider the case where $\dim M = 2$ and $\rho(f)$ hyperbolic. Our goal in the sequel is to show that in this case, too, (M, g) has constant curvature.

The spectrum of K_f consists of λ , and one constant, say λ_- .

7.0.7. Warped product structure. Let \mathcal{F}_- and \mathcal{F}_λ the two one dimensional foliations tangent to the eigenspaces E_- and E_λ . They are regular foliations on $M - \mathcal{S}_-$ (where $\mathcal{S}_- = \{x/\lambda(x) = \lambda_-\}$).

Recall that Dini normal form says that for two projectively equivalent metrics g and \bar{g} on a surface have the following form:

$$g = (X(x) - Y(y))(dx^2 + dy^2), \quad \bar{g} = \left(\frac{1}{Y(y)} - \frac{1}{X(x)}\right)\left(\frac{dx^2}{X(x)} + \frac{dy^2}{Y(y)}\right)$$

near any point where the $(1, 1)$ -tensor L defined by $\bar{g}(\cdot, \cdot) = g(\frac{L^{-1}}{\det L} \cdot, \cdot)$, has simple eigenvalues. In fact, $X(x)$ and $Y(y)$ are the eigenvalues of $L(x, y)$, and the co-ordinates are adapted to eigenspaces.

From this normal form one deduces that $(\mathcal{F}_\lambda, \mathcal{F}_-)$ determines a warped product (for g as well as \bar{g}), that is, there are coordinates (r, θ) , where $\frac{1}{\partial r}$ (resp. $\frac{1}{\partial \theta}$) is tangent to \mathcal{F}_λ (resp. \mathcal{F}_-), and the metric has the form $dr^2 + \delta(r)d\theta^2$ (see [34] for more information on warped products). Indeed here $Y(y) = \lambda_-$, and hence $g = (X(x) - \lambda_-)dx^2 + (X(x) - \lambda_-)dy^2$, and then change coordinates according to $dr^2 = (X(x) - \lambda_-)dx^2$, $\theta = y$.

We deduce in particular that the leaves of \mathcal{F}_λ are geodesic in (M, g) .

7.0.8. Topology. By 6.3.2, $D_x f$ is contracting away from $\mathcal{S}_+ = \{x/\lambda(x) = \lambda_+\}$.

Let $c \in]\lambda_-, \lambda_+[$ and $M_c = \{x/\lambda(x) \leq c\}$. It is a codimension 0 compact submanifold with boundary the level $\lambda^{-1}(c)$, for c generic. In particular, it has a finite number of connected components. Since λ is decreasing with f (§6.2), f preserves M_c : $f(M_c) \subset M_c$. Taking a power of f , we can assume it preserves each component of M_c . On such a component, say M_c^0 , f contracts the Riemannian metric, and hence also its generated distance. It follows that f has a unique fixed point $x_0 \in M_c^0$. The M_c^0 's, for c decreasing to λ_- , is a decreasing family converging to x_0 . It follows that these M_c 's are topological discs, their boundaries $\lambda^{-1}(c)$ are circles surrounding x_0 , and finally x_0 is the unique point in M_c^0 with $\lambda(x) = \lambda_-$.

It is a general fact on tensors of $\mathcal{L}(M, g)$, an eigenfunction is constant along leaves tangent to the eigenspaces associated to the other eigen-functions. In our case, λ is constant on the \mathcal{F}_- leaves, equivalently, leaves of \mathcal{F}_- are levels of λ . So, these leaves are circles surrounding

x_0 . Those of \mathcal{F}_λ are orthogonal to these circles, and hence they are nothing but the geodesics emanating from x_0 . Thus, \mathcal{F}_λ and \mathcal{F}_- have x_0 as a unique singularity in M_c^0 .

7.0.9. Geometry. We infer from the above analysis that the polar coordinates around x_0 gives rise to a warped product structure, that is, the metric on these coordinates (r, θ) , has the form: $dr^2 + \delta(r)d\theta^2$ (in the general case δ depends rather on (r, θ)).

At x_0 , $D_{x_0}f$ is a similarity with coefficient λ_- .

Observe next that, from the form of the metric, rotations $\theta \rightarrow \theta + \theta_0$ are isometries. Composing f with a suitable rotation, we can assume f is a “pure homothety”, i.e. it fixes individually each geodesic emanating from x_0 . Thus f acts only at the r -level: $f(r, \theta) = f(r)$.

Now, the idea is to construct a higher dimensional example with the same ingredients, e.g. δ and f . Precisely, consider the metric $dr^2 + \delta(r)d\Omega^2$, where $d\Omega^2$ is the standard metric on the sphere \mathbb{S}^N . We let f acts by $f(r, \Omega) = f(r)$.

One verifies that f is projective. Indeed, $\text{SO}(N+1)$ acts isometrically and commutes with f , and any geodesic is contained in a copy of our initial surface.

This new f has the same dynamical behaviour as the former one, and one proves as in §6 that the projective Weyl tensor of this new metric vanishes and it has therefore a constant sectional curvature. The same is true for our initial surface.

□

8. THE KÄHLER CASE: PROOF OF THEOREM 1.5

Let $F(N, b)$ denote the simply connected Hermitian space of dimension N and constant holomorphic sectional curvature b . Calabi proved (in his thesis) the following striking fact:

Theorem 8.1 (Calabi [7]). *Let M be a Kähler manifold (not necessarily complete) and $f : M \rightarrow F(N, b)$ a holomorphic isometric immersion. Then, f is rigid in the sense that any other immersion f' is deduced from f by composing with an element of $\text{Iso}(F(N, b))$ (this element is unique if the image of f is not contained in a totally geodesic subspace of $F(N, b)$). In particular, f is equivariant with respect to some faithful representation $\text{Iso}(M) \rightarrow \text{Iso}(F(N, b))$.*

As for holomorphic isometric immersions between space forms, one deduces (for more information, see for instance in [30, 16, 10]):

Theorem 8.2. – *The Kähler Euclidean space \mathbb{C}^d can not embed holomorphically isometrically in a projective space $\mathbb{P}^N(\mathbb{C})$ (a radially simple example of this is the situation of a holomorphic vector field; it does not act isometrically, in particular its orbits are not metrically homogeneous).*

– *Up to ambient isometry, the holomorphic homothetic embeddings between projective spaces are given by Veronese maps: $v_k : (\mathbb{P}^d(\mathbb{C}), g_{FS}) \rightarrow (\mathbb{P}^N(\mathbb{C}), \frac{1}{k}g_{FS})$, $N = \binom{d+k}{k} - 1$, $v_k : [X_0, \dots, X_d] \rightarrow [\dots X^I \dots]$, where X^I ranges over all monomials of degree k in X_0, \dots, X_d .*

Proof of Theorem 1.5. This will follow from the Kähler version of projective Lichnerowicz conjecture (stated as in Theorem 1.3) together with the following fact.

Fact 8.3. *Let $(M^d, g_{SF|_M})$ be a submanifold of $(\mathbb{P}^N(\mathbb{C}), g_{SF})$, then $\text{Aff}(M^d, g_{SF|_M})/\text{Iso}(M^d, g_{SF|_M})$ is finite (vaguely bounded by $\frac{n}{2}!$).*

Proof. This is a standard idea (see for instance [19, 35]), the unique special fact we use here is that, by Calabi Theorem, the universal cover has no flat factor in its De Rham decomposition. Thus \tilde{M} is a product $\tilde{M}_1 \times \dots \times \tilde{M}_m$ of irreducible Kähler manifolds.

The holonomy group $\text{Hol}^{\tilde{M}}$ equals the product $\text{Hol}^{\tilde{M}_1} \times \dots \times \text{Hol}^{\tilde{M}_m}$. An affine transformation \tilde{f} commutes with Hol and hence preserves the De Rham splitting. Taking a power, we can assume that \tilde{f} actually preserves each factor, and we will thus prove that \tilde{f} is isometric. Since \tilde{M}_i is irreducible, \tilde{f} induces a homothety on it, say of distortion c . If $c \neq 1$, then f or f^{-1} is contracting with respect to the distance of \tilde{M}_i . In this case, \tilde{f} will have a (unique) fixed point in \tilde{M}_i . However, \tilde{f} preserves the Riemann curvature tensor $R(X, Y)Z$ of \tilde{M}_i . But being invariant by a contraction (or a dilation), this tensor must vanish, that is \tilde{M}_i is flat, contradicting the fact that it is irreducible. Therefore, $c = 1$, that is \tilde{f} is isometric. \square

Remarks 8.4.

1. By equivariance, Segre maps $\mathbb{P}^m(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C}) \rightarrow \mathbb{P}^{mn+1}(\mathbb{C})$ are homothetic. In particular, some $(\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}), \frac{1}{k}(g_{FS} \oplus g_{FS}))$ can be embedded in some $(\mathbb{P}^N(\mathbb{C}), g_{FS})$. By composing Veronese and Segre maps, one can also realize some metrics $(\mathbb{P}^1(\mathbb{C}), \frac{1}{k}g_{FS}) \times (\mathbb{P}^1(\mathbb{C}), g_{FS})$.

2. In fact, it turns out that for M a submanifold of $\mathbb{P}^N(\mathbb{C})$, De Rham decomposition applies to M itself; that is the splitting of \tilde{M} descends to a one of M . I am indebted to A. J. Di Scala for giving me a proof of that using Calabi rigidity. Indeed, this rigidity has the following amazing corollary: if M is holomorphically isometrically embedded in $\mathbb{P}^N(\mathbb{C})$, then neither a cover nor a quotient of it can be embedded so. Now, the De Rham splitting of \tilde{M} gives an immersion into products of projective spaces. Segre map is isometric form this product to one big projective space, which gives us another holomorphic isometric immersion of \tilde{M} . But this must coincide with the immersion given by the universal cover $\tilde{M} \rightarrow M$. This implies that the De Rham decomposition is defined on M itself.

9. FACTS ON THE PSEUDO-RIEMANNIAN CASE: PROOF OF THEOREM 1.7

Let (M, g) be a compact pseudo-Riemannian manifold with projective degree of mobility $\dim \mathcal{L}(M, g) = 2$, such that $\text{Proj}(M, g)/\text{Aff}(M, g)$ is infinite. Consider $\rho : \text{Proj}(M, g) \rightarrow \text{GL}_2(\mathbb{R})$.

Denote $G = \rho(\text{Proj}(M, g))$. Theorem 1.7 says that, up to finite index, $\ker \rho = \text{Iso}(M, g) = \text{Aff}(M, g)$, and G lies in a one parameter group.

9.1. “Projective linear” action of $\text{Proj}(M, g)$. So far, we singled out an element $f \in \text{Proj}(M, g)$ and associate to it a homography A_f acting on $\bar{\mathbb{R}}$. It turns out that this A_f is nothing but that corresponding to the (projective) action of $\rho(f)$ on the projective space of $\mathcal{L}(M, g)$, identified to $\mathbb{P}^1(\mathbb{R})$, via the basis $\{K = K_f, I\}$.

Indeed, the choose of the basis $\{K, I\}$, say co-ordinates (k, i) , allows one to identify $\mathbb{P}(\mathcal{L}(M, g))$ with $\mathbb{P}^1(\mathbb{R})$. In the affine chart $[k : i] \in \mathbb{P}^1(\mathbb{R}) \rightarrow z = \frac{k}{i}$, the projective action of $\rho(f)$ is $z \rightarrow \frac{\alpha z + \beta}{\gamma z + \delta}$, where α and β are defined by $\rho(f)K = \alpha K + \beta I$, as in §4.

Now, we let the whole group $\text{Proj}(M, g)$ act by means of ρ on the projective space, and in fact complex one. More precisely, let

$$\Phi : \text{Proj}(M, g) \rightarrow \text{PGL}(\mathcal{L}(M, g) \otimes \mathbb{C})$$

be the action associated to ρ on $\mathbb{P}(\mathcal{L}(M, g) \otimes \mathbb{C})$, the projective space of the complexification of $\mathcal{L}(M, g)$.

The degeneracy set \mathcal{D} is complexified as

$$\mathcal{D}^{\mathbb{C}} = \{L \in \mathbb{P}^1(\mathcal{L}(M, g) \otimes \mathbb{C}), L \text{ not an isomorphism of } TM \otimes \mathbb{C}\}$$

The proof of the following fact is similar to that of Fact 4.1.

Fact 9.1. *Let f be any element of $\text{Proj}(M, g)$ with $K_f \neq -\pm I$, then $\mathcal{D}^{\mathbb{C}}$ can be computed by means of K_f as follows. Under the identification of $\mathbb{P}(\mathcal{L}(M, g) \otimes \mathbb{C})$ with $\mathbb{P}^1(\mathbb{C})$ via the basis $\{K_f, I\}$, the set $\mathcal{D}^{\mathbb{C}}$ corresponds to the range of the spectrum mapping of K_f : $x \in M \rightarrow Sp^{K_f}(x) = \text{Spectre of } K_f(x) \subset \mathbb{C}$.*

9.2. First steps in the proof of Theorem 1.7.

9.2.1. By Fact 3.2, the projection of G on its image in $\text{PGL}_2(\mathbb{R})$ has finite index. In fact, since we are interested in objects up to finite index, for simplicity seek, we will argue as if G is contained in $\text{SL}_2(\mathbb{R})$.

9.2.2. *The Kernel of ρ .* Let $h \in \text{Aff}(M, g)$, then K_h has constant eigenvalues. It follows that K_h has the form aI , since otherwise it generates together with I the whole $\mathcal{L}(M, g)$, and hence K_f will have constant eigenvalues for any f , contradicting the fact that $\text{Proj}(M, g) \supsetneq \text{Aff}(M, g)$. Now, $\rho(h)L = h_*LK_h$, $K_h = aI$, and thus I is fixed by $\rho(h)$, which is hence parabolic. But, on $\mathbb{P}^1(\mathbb{C})$, $\Phi(h)$ preserves another closed invariant set $\mathcal{D}^{\mathbb{C}}$, far away from its fixed point I . Hence, $\rho(h) = Id$.

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9.2.3. If $\mathcal{D}^{\mathbb{C}} \cap \bar{\mathbb{R}} = \emptyset$, then $\mathcal{D}^{\mathbb{C}}$ is contained in $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$ which is a union of two copies of a hyperbolic upper plane. In general, the group of isometries preserving a compact subset of the hyperbolic plane is compact and contained in a conjugate of $\text{SO}(2)$.

9.2.4. If $\mathcal{D}^{\mathbb{C}} \cap \bar{\mathbb{R}} \neq \emptyset$, then \bar{G} is a proper subgroup of $\text{SL}_2(\mathbb{R})$, and its identity component \bar{G}^0 is a connected subgroup of it. Thus, up to a conjugacy, \bar{G}^0 is either trivial, a one parameter group or the affine group $\text{Aff}(\mathbb{R})$.

Now, $\bar{G}^0 = 1$ means exactly that G is discrete.

Actually, \bar{G}^0 can not be an elliptic one parameter group. Indeed this implies $\mathcal{D}^{\mathbb{C}} \cap \bar{\mathbb{R}} = \{0\}$.

Since G is contained in the normalizer of \bar{G}^0 , in all cases where G is not discrete, G is, up to conjugacy, contained in $\text{Aff}(\mathbb{R})$.

9.3. Cases.

9.3.1. *Elliptic case.* If all elements of G are elliptic, necessarily G is discrete. But a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ with only elliptic elements is conjugate to a finite subgroup of $\mathrm{SO}(2)$ (Indeed, all G -elements are of torsion type. Selberg Theorem says that any finitely generated subgroup H has a torsion free subgroup of finite index. Hence H is finite. Unless trivial, H has a unique fixed point z_0 in the hyperbolic plane. By uniqueness, z_0 will be fixed by any finitely generated subgroup of G bigger than H . Hence z_0 is fixed by G).

9.3.2. *Parabolic case.* The range of the spectrum mapping Sp^{K_f} touches the real projective line $\mathbb{P}^1(\mathbb{R})$ exactly at F_f the unique fixed point of $\Phi(f)$. Therefore, the upper part of this range lies in a unique minimal horoball, i.e. a disk tangent to $\mathbb{P}^1(\mathbb{R})$ at F_f . The full range is contained in the union of this horoball with its conjugate. Therefore Φ preserves this unions of horoballs, however, the subgroup of $\mathrm{PGL}_2(\mathbb{C})$ preserving such a horoball is exactly the one parameter group generated by $\Phi(f)$.

9.3.3. *Hyperbolic case.* Now, if $\Phi(f)$ has fixed points F_f^{-1} and F_f^+ . Then the spectrum range meets the real projective line in a subset of the segment $[F_f^{-1}, F_f^+]$. Here the range spectrum is contained in $B \cup B'$, where B is the smallest disc containing the part of spectrum on the upper half plane, and B' is its conjugate (i.e. image under $z \rightarrow \bar{z}$). As previously, the subgroup of $\mathrm{PSL}_2(\mathbb{C})$ preserving $B \cup B'$, fixes $\{F_f^{-1}, F_f^+\}$, and is up to a switch (i.e. an involution inverting the F_f^\pm), the hyperbolic one parameter group generated by $\Phi(f)$.

9.3.4. We have thus proved that neglecting finite index objects, $\rho(\mathrm{Proj}(\mathcal{M}, \mathfrak{g}))$ lies in a one parameter group, which completes the proof of Theorem 1.7. \square

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